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Fuzzy Sets

- Basic definitions
- Aggregation operators
- Extension principle

Crisp sets

- Collection of definite, well-definable objects (elements) to form a whole.

Representation of sets:

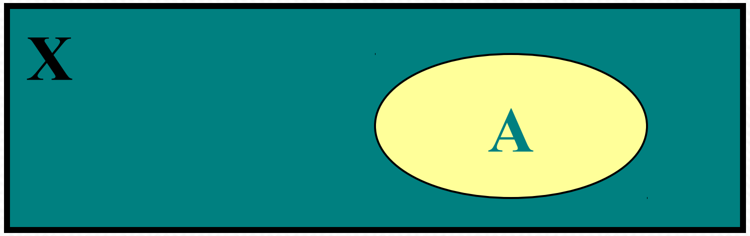
- list of all elements

  $A=\{x_1, \ldots, x_n\}, x_j \in X$

- elements with property P

  $A=\{x|x \text{ satisfies } P\}, x \in X$

- Venn diagram



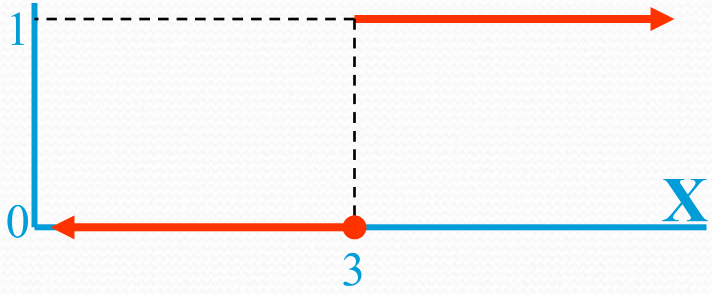
- characteristic function

  $f_A: X \to \{0,1\},$

  $f_A(x) = 1, \Leftrightarrow x \in A$

  $f_A(x) = 0, \Leftrightarrow x \notin A$

Real numbers larger than 3:



Fuzzy sets

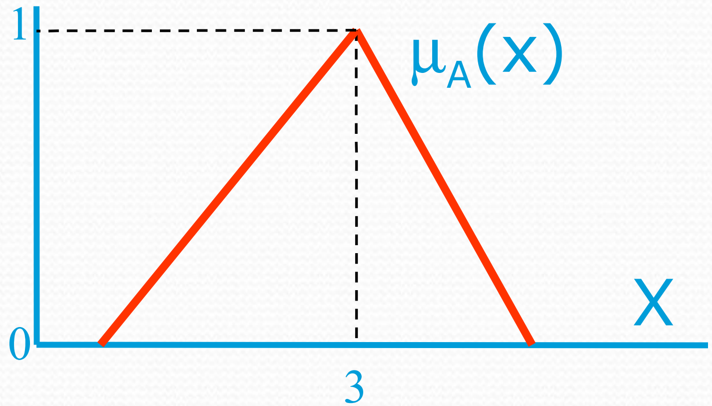
- Sets with fuzzy, gradual boundaries (Zadeh 1965)
- A fuzzy set A in X is characterized by its membership function $\mu_A: X \rightarrow [0,1]$

A fuzzy set A is completely determined by the set of ordered pairs

$A=\{(x,\mu_A(x))| x \in X\}$

X is called the *domain* or *universe of discourse*

Real numbers about 3:



Fuzzy sets on discrete universes

- Fuzzy set C = "desirable city to live in"

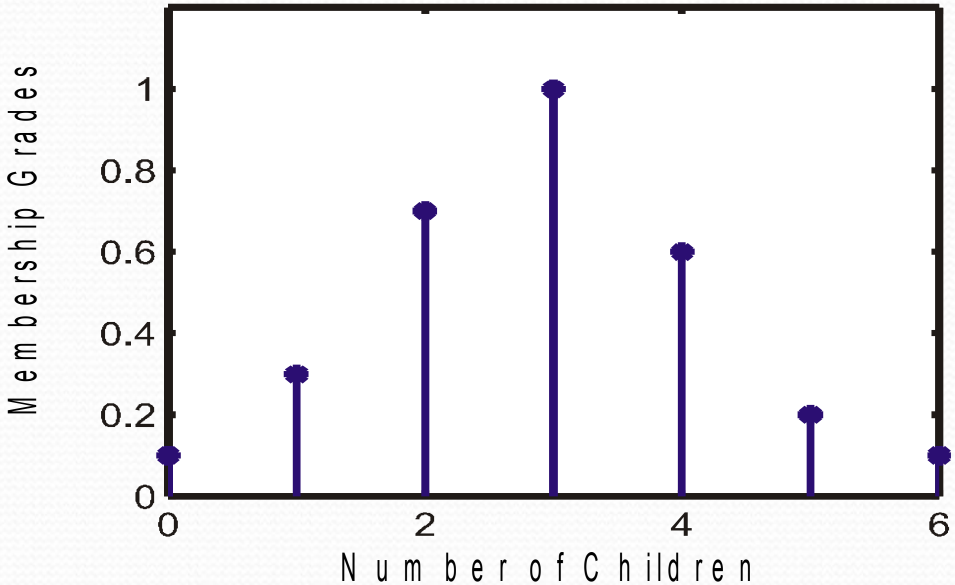
  X = {SF, Boston, LA} (discrete and non-ordered)

  C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

- Fuzzy set A = "sensible number of children"

  X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)

  A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



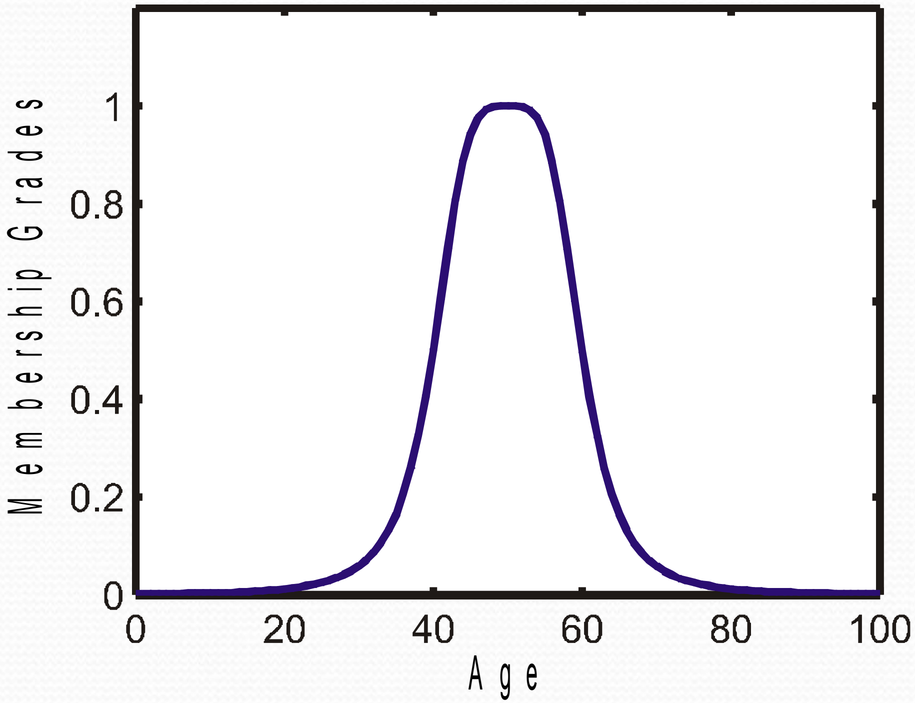
Fuzzy sets on continuous universes

- Fuzzy set B = "about 50 years old"

  X = Set of positive real numbers (continuous)

  B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^2}$$



Notation

Many texts (especially older ones) do not use a consistent and clear notation

X is discrete

$$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

$$A = \sum_{x_i \in X} \mu_A(x_i)\ x_i$$

X is continuous

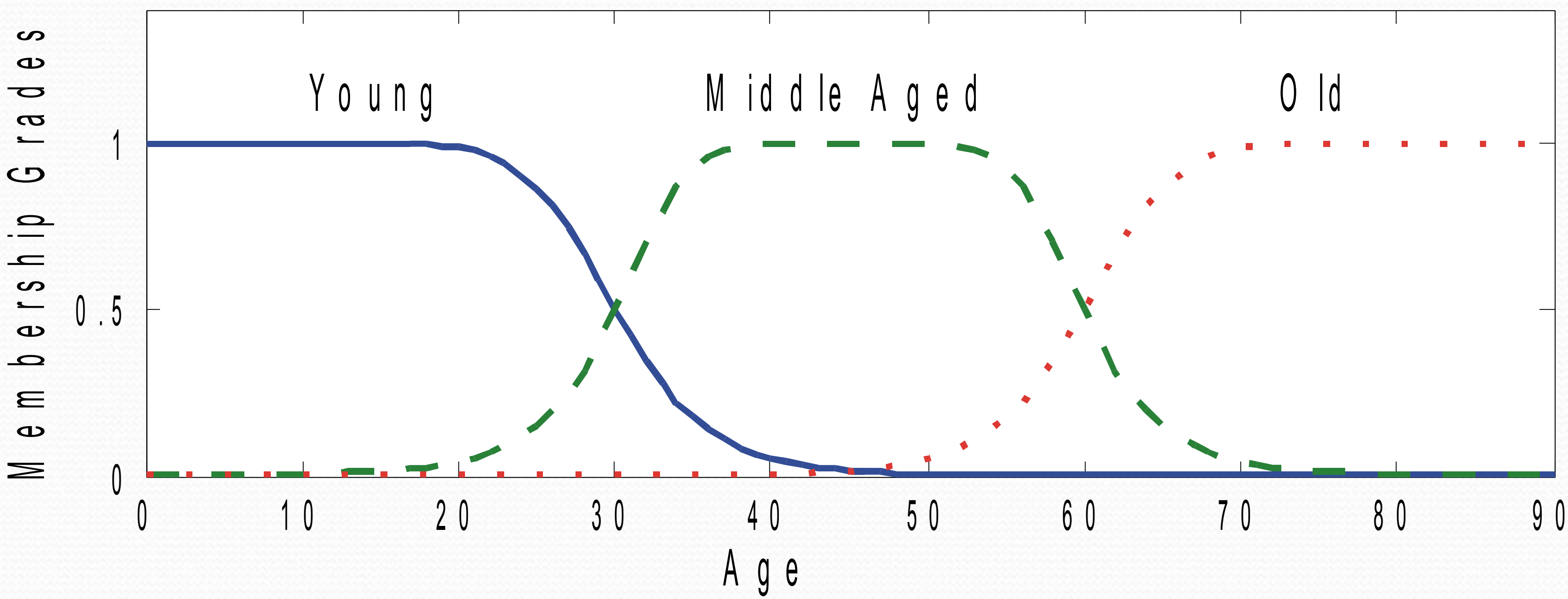
$$A = \int_X \mu_A(x) / x$$

$$A = \int_X \mu_A(x)\ x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

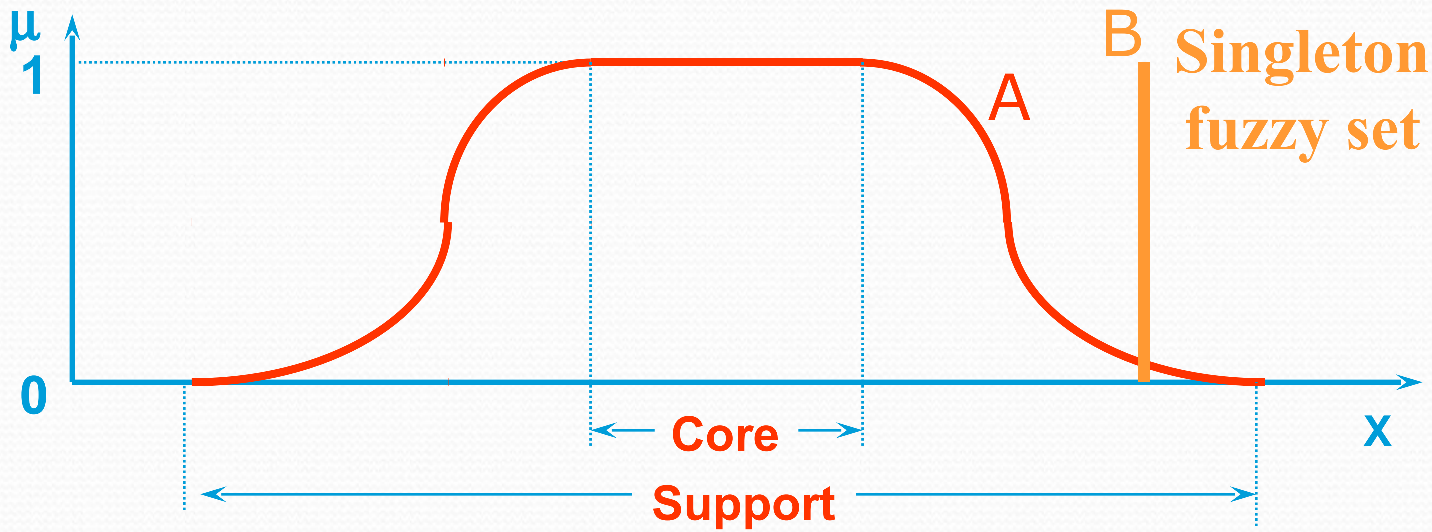
Fuzzy partition

Fuzzy partition formed by the linguistic values "young", "middle aged", and "old":



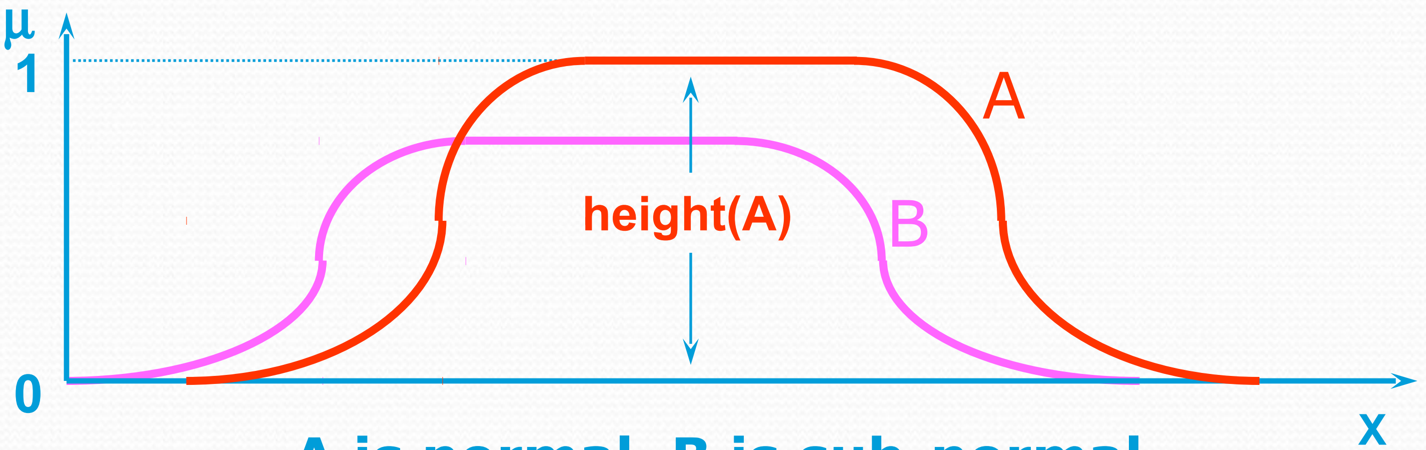
Support, core, singleton

- The *support* of a fuzzy set A in X is the crisp subset of X whose elements have non-zero membership in A: $supp(A) = \{x \in X \mid \mu_A(x) > 0\}$
- The *core* of a fuzzy set A in X is the crisp subset of X whose elements have membership 1 in A: $core(A) = \{x \in X \mid \mu_A(x) = 1\}$



Normal fuzzy sets

- The *height* of a fuzzy set A is the maximum value of $\mu_A(x)$
- A fuzzy set is called *normal* if its height is 1, otherwise it is called *sub-normal*

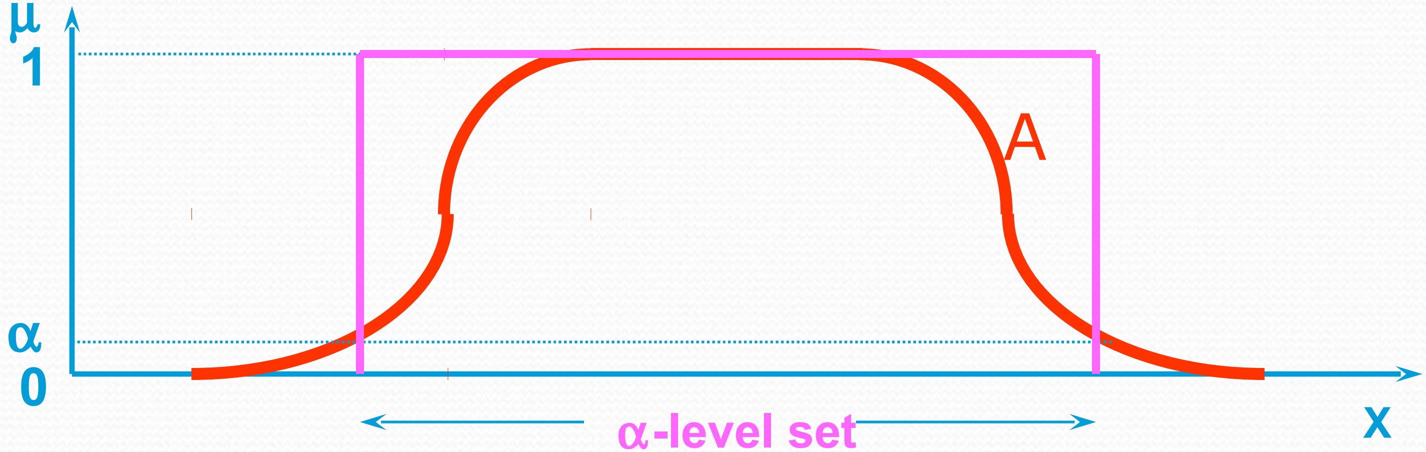


A is normal, B is sub-normal

α-cut of a fuzzy set (level set)

- An α-level set of a fuzzy set A of X is a crisp set denoted by A_α and defined by

$$A_\alpha = \{ x \in X \mid \mu_A(x) \geq \alpha \}, \quad \alpha > 0$$

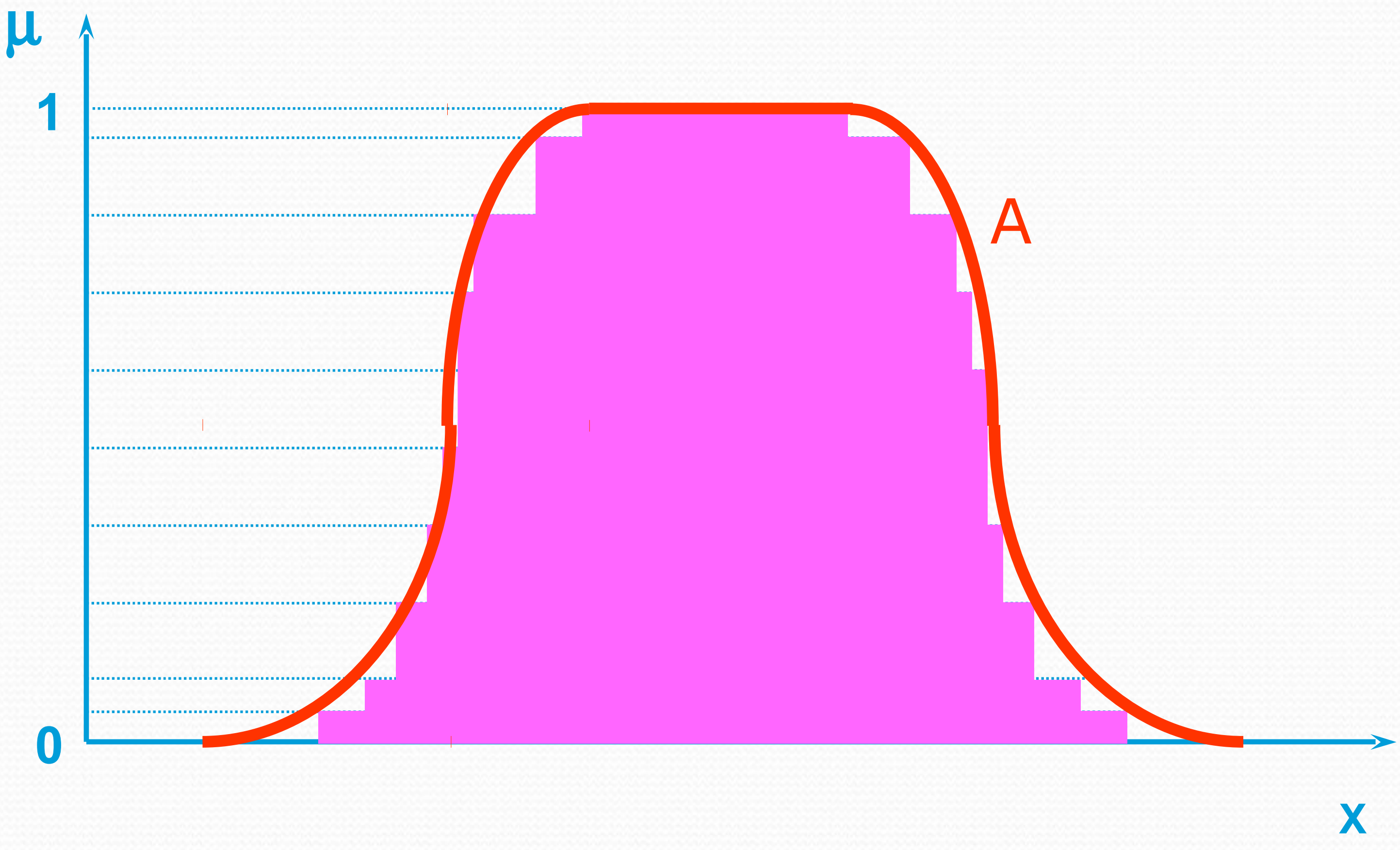


"Resolution principle"

Every fuzzy set A can be uniquely represented as a collection of α-level sets according to

$$\mu_A(x) = \sup_{\alpha \in [0,1]} [\alpha \mu_{A_\alpha}(x)]$$

Resolution principle



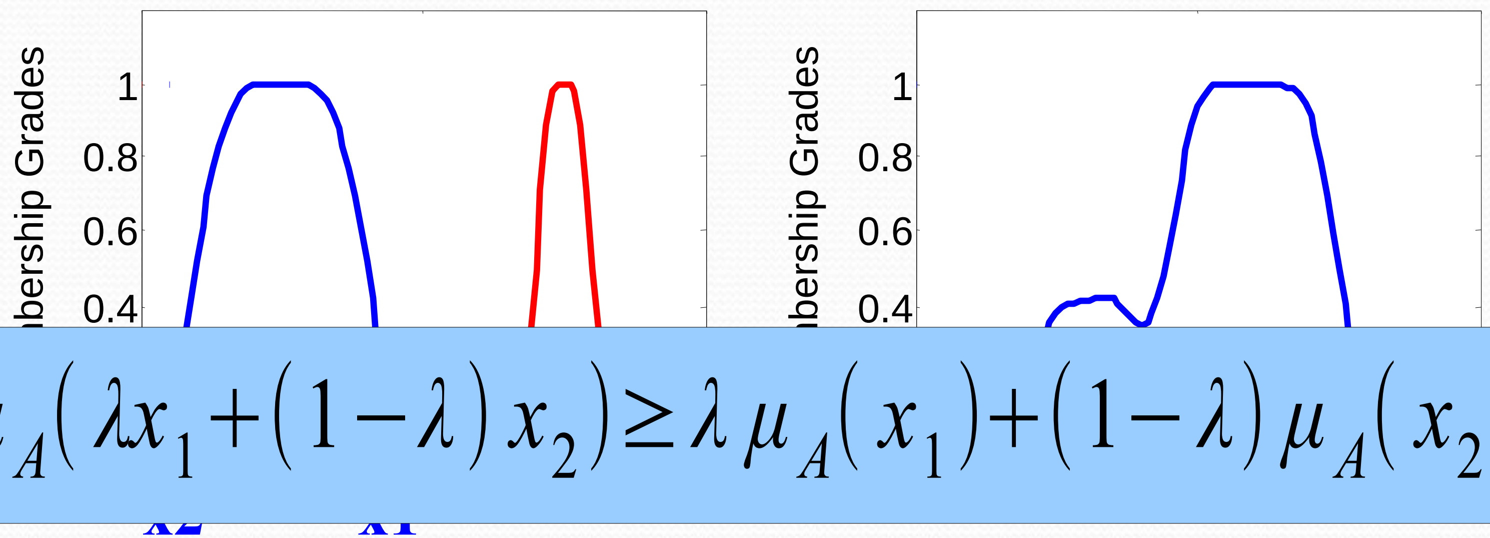
Convexity of fuzzy sets

A fuzzy set A is convex if for any λ in [0, 1] and any x1, x2 in the support set,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, *A* is convex if all its α-cuts are convex

(a) Two Convex Fuzzy Sets (b) A Nonconvex Fuzzy Set



$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \lambda \mu_A(x_1) + (1-\lambda)\mu_A(x_2)$$

Symmetry, left-right open

A fuzzy set A is symmetric if its MF is symmetric around a certain point $x = c$, i.e.

$$\mu_A(c + x) = \mu_A(c - x), \quad \forall x \in X$$

A fuzzy set A is open left if $\lim_{x \to -\infty} \mu_A(x) = 1$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

A fuzzy set A is open right if $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 1$

A fuzzy set A is closed if $\lim_{x \to -\infty} \mu_A(x) = 0$ and $\lim_{x \to +\infty} \mu_A(x) = 0$

Fuzzy number, width

- A fuzzy number is a fuzzy set in the line of real numbers that is normal and convex
- Fuzzy numbers are the most basic types of fuzzy sets (convex and normal)
- For a normal and convex fuzzy set A, the width is defined as the area under the membership function
- If the membership function is trapezoidal,

$$\text{width}(A) = |x_2 - x_1|$$

$$\text{where } \mu_A(x_1) = \mu_A(x_2) = 0.5$$

Set theoretic operations **(Specific case)**

- Subset:

$$A \subseteq B \Leftrightarrow \mu_A \leq \mu_B$$

- Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

- Union:

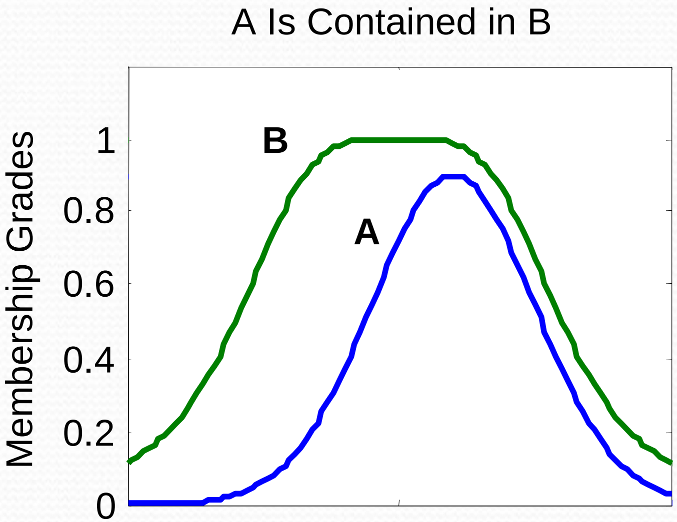
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

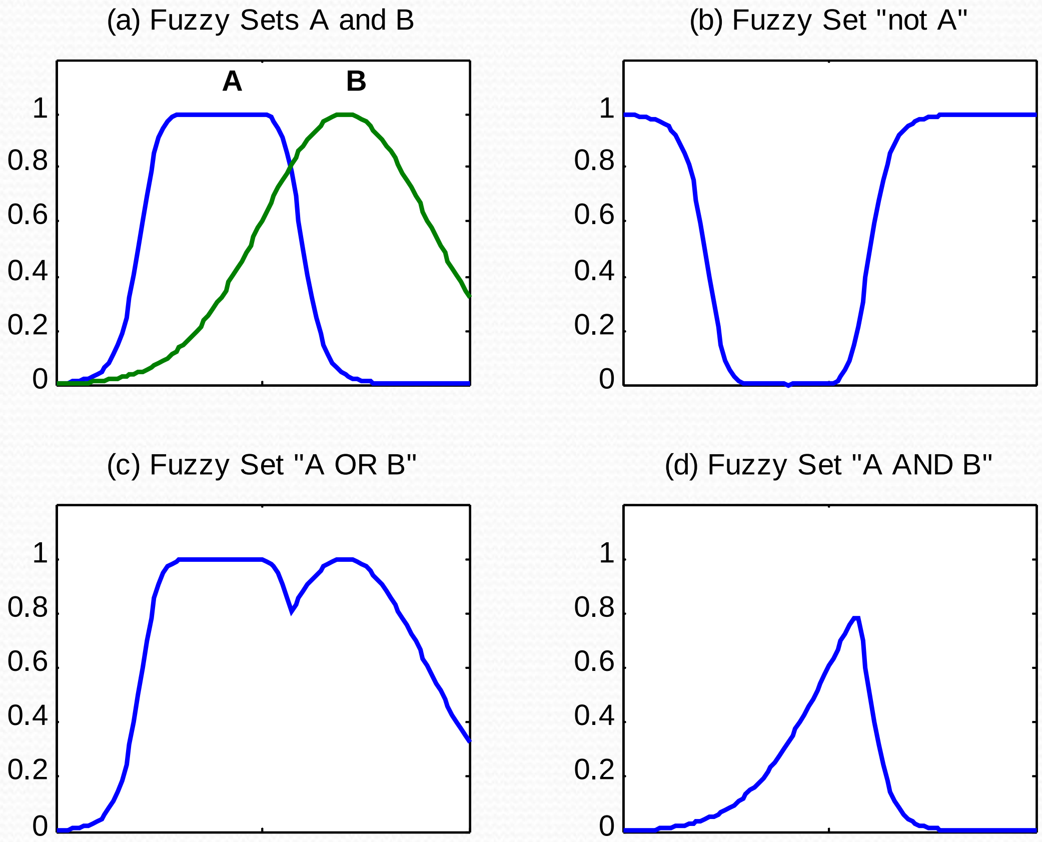
- Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set theoretic operations

$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$$





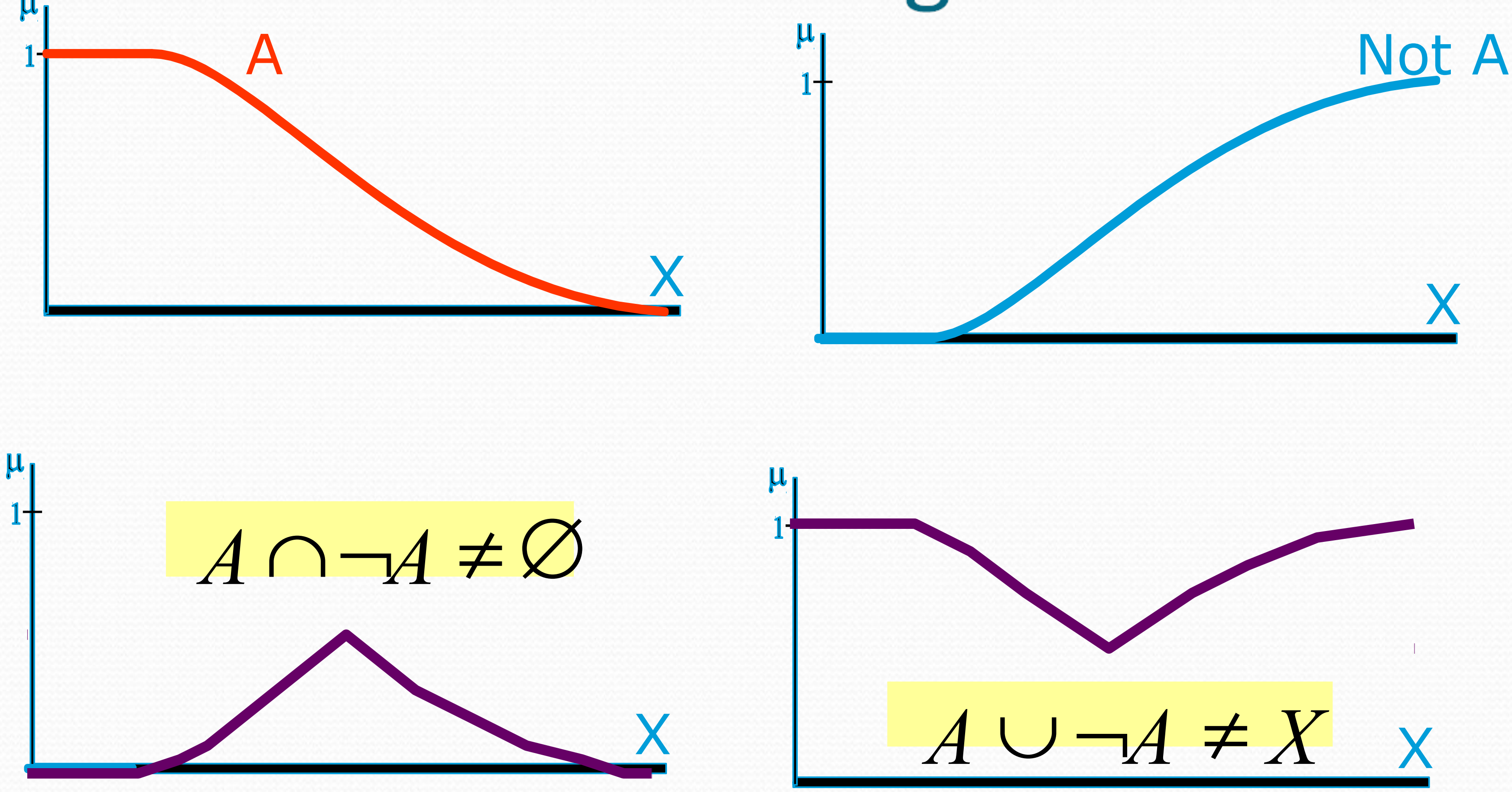
Average

The average of fuzzy sets A and B in X is defined by

$$\mu_{(A+B)/2}(x) = \frac{\mu_A(x) + \mu_B(x)}{2}$$

Note that the classical set theory does not have averaging as a set operation. This is an extension provided by the fuzzy set approach.

Combinations with negation



Note: De Morgan laws do hold in fuzzy set theory!

Cartesian product

- Cartesian product of fuzzy sets A and B is a fuzzy set in the product space X x Y with membership

$$\mu_{A\times B}(x,y)=\min(\mu_A(x),\mu_B(y))$$

- Cartesian co-product of fuzzy sets A and B is a fuzzy set in the product space X x Y with membership

$$\mu_{A+B}(x,y)=\max(\mu_A(x),\mu_B(y))$$

Membership Function formulation

Triangular MF:

$$trimf(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

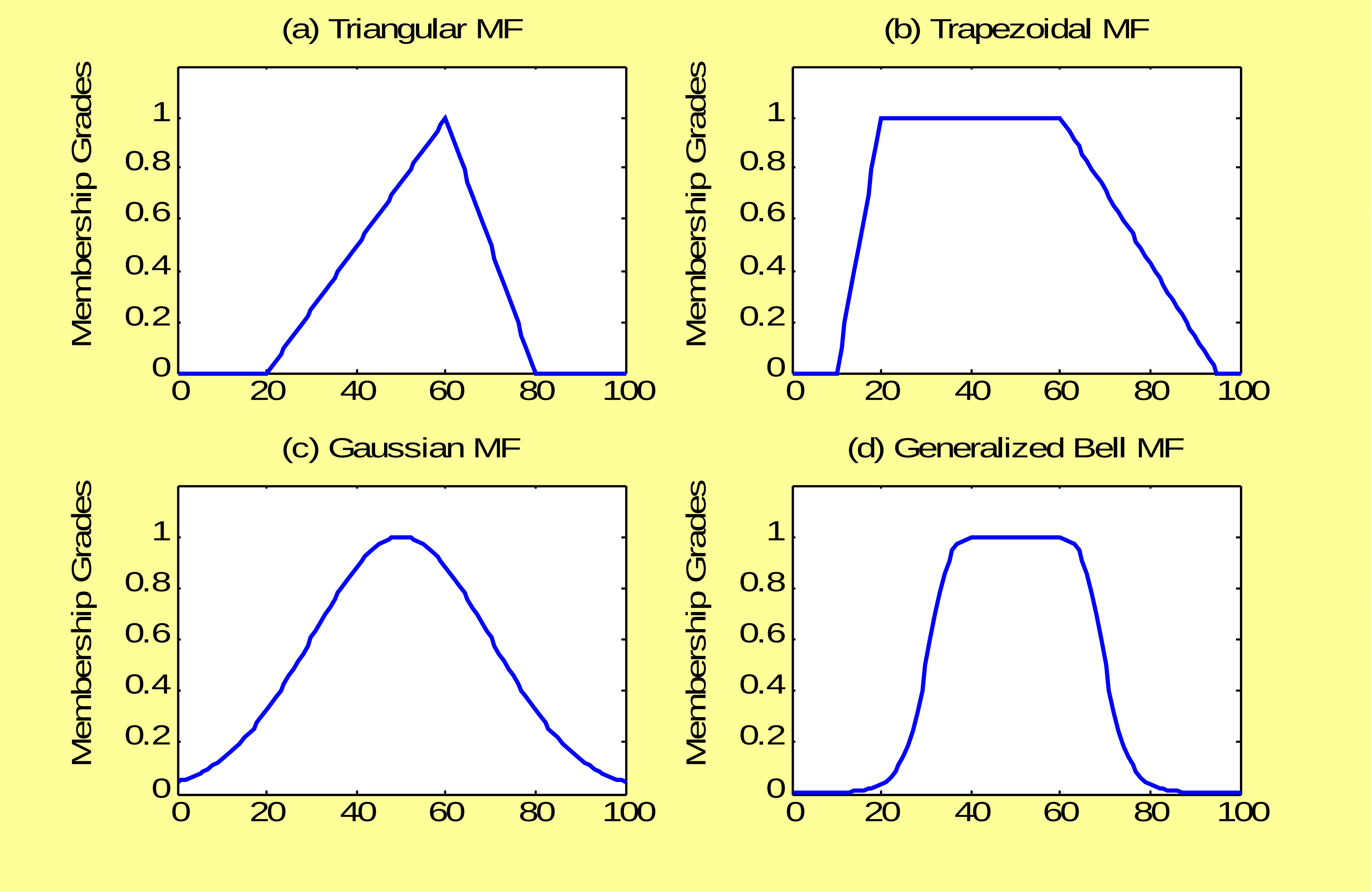
Gaussian MF:

$$gaussmf(x;a,b) = e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$

Generalized bell MF:

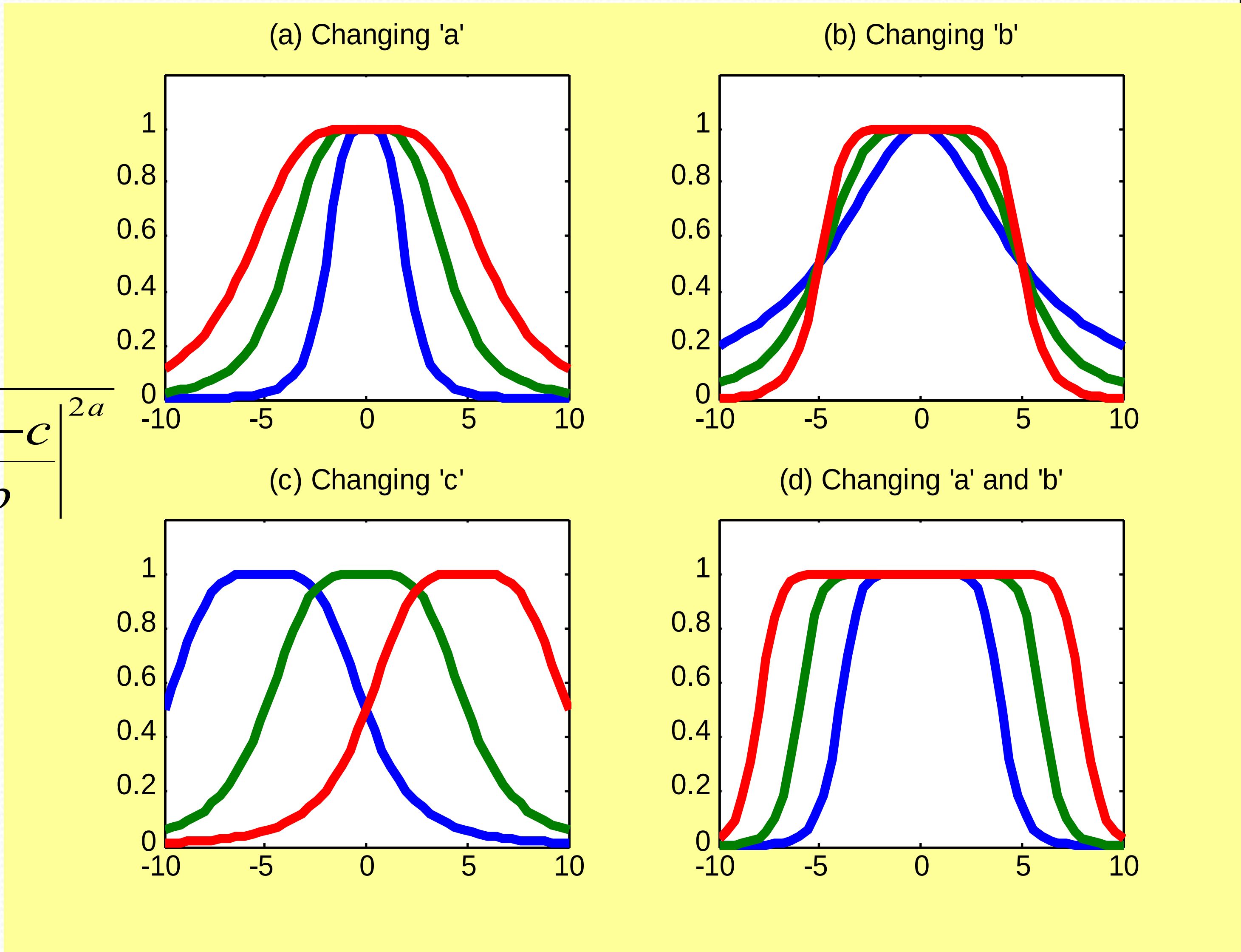
$$gbellmf(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{b}\right|^{2a}}$$

MF formulation



Generalized bell MF

$$\mu(x) = \frac{1}{1 + \left| \frac{x - c}{b} \right|^{2a}}$$



MF formulation

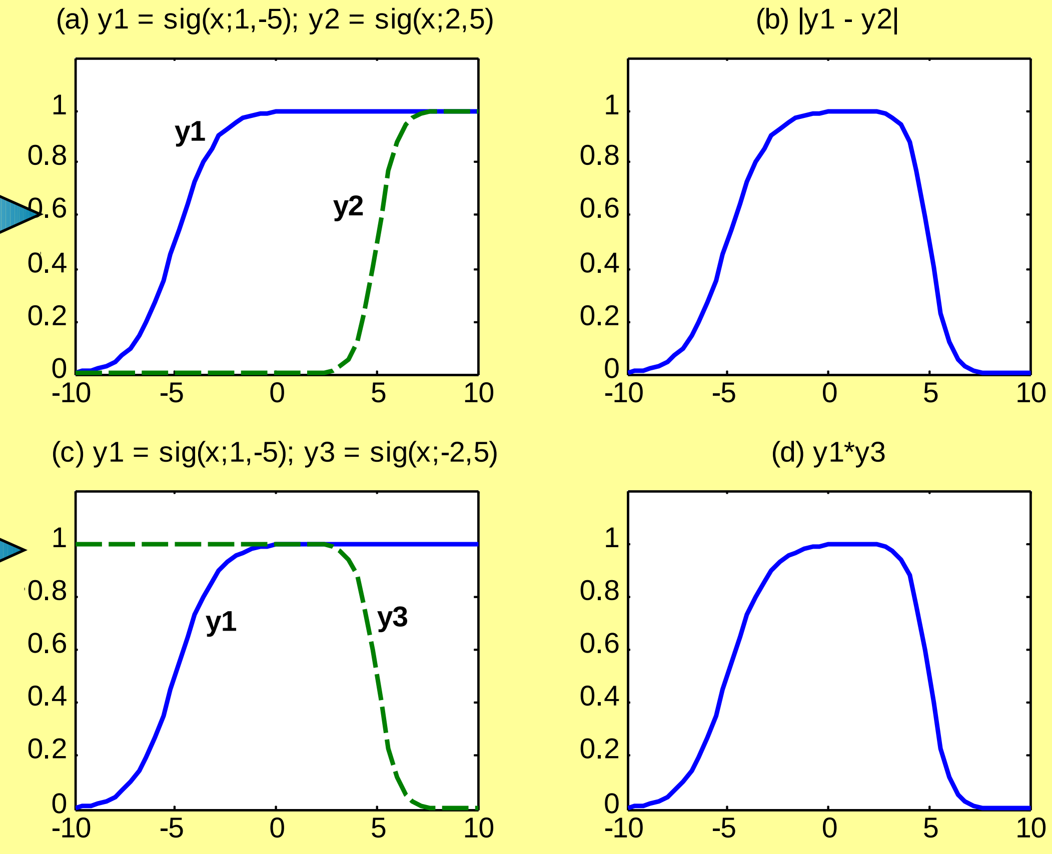
Sigmoidal MF:

$$\text{sigmf}(x;a,b,c) = \frac{1}{1+e^{-a(x-c)}}$$

Extensions:

Abs. difference of two sig. MF

Product of two sig. MF



MF formulation

L/R type function:

- $F_L(0)=1$
- $F_L(x)<1$ **for all** $x>0$
- $F_L(x)=0$ **for** $x \rightarrow$ ***infinity***

$$F_L(x) = \sqrt{\max(0, 1 - x^2)}$$

$$F_R(x) = \exp(-|x|^3)$$

MF formulation

L-R MF:

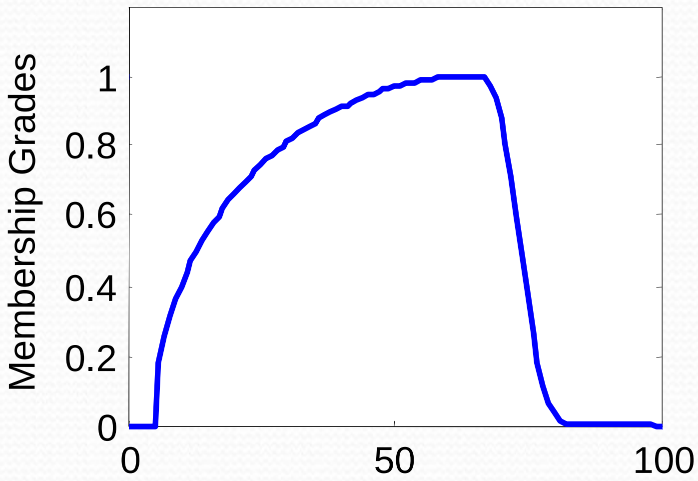
$$LR(x;c,\alpha,\beta)=\begin{cases} F_L\left(\frac{c-x}{\alpha}\right), x<c \\ F_R\left(\frac{x-c}{\beta}\right), x\geq c \end{cases}$$

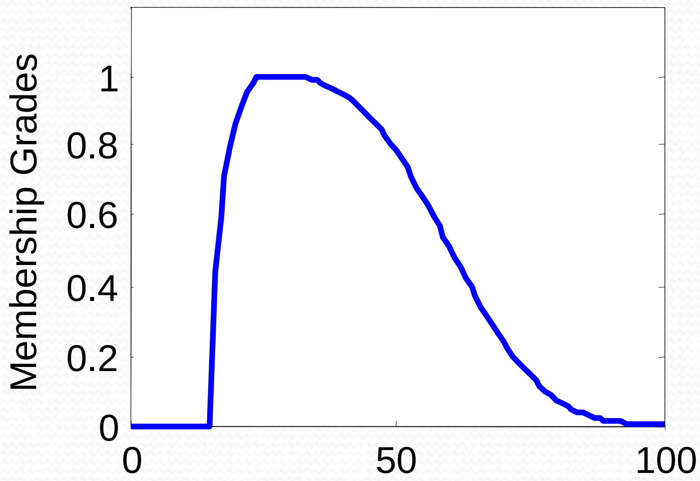
Example: $F_L(x)=\sqrt{\max(0,1-x^2)}$ $F_R(x)=\exp(-|x|^3)$

c=65

α=60

β=10





c=25

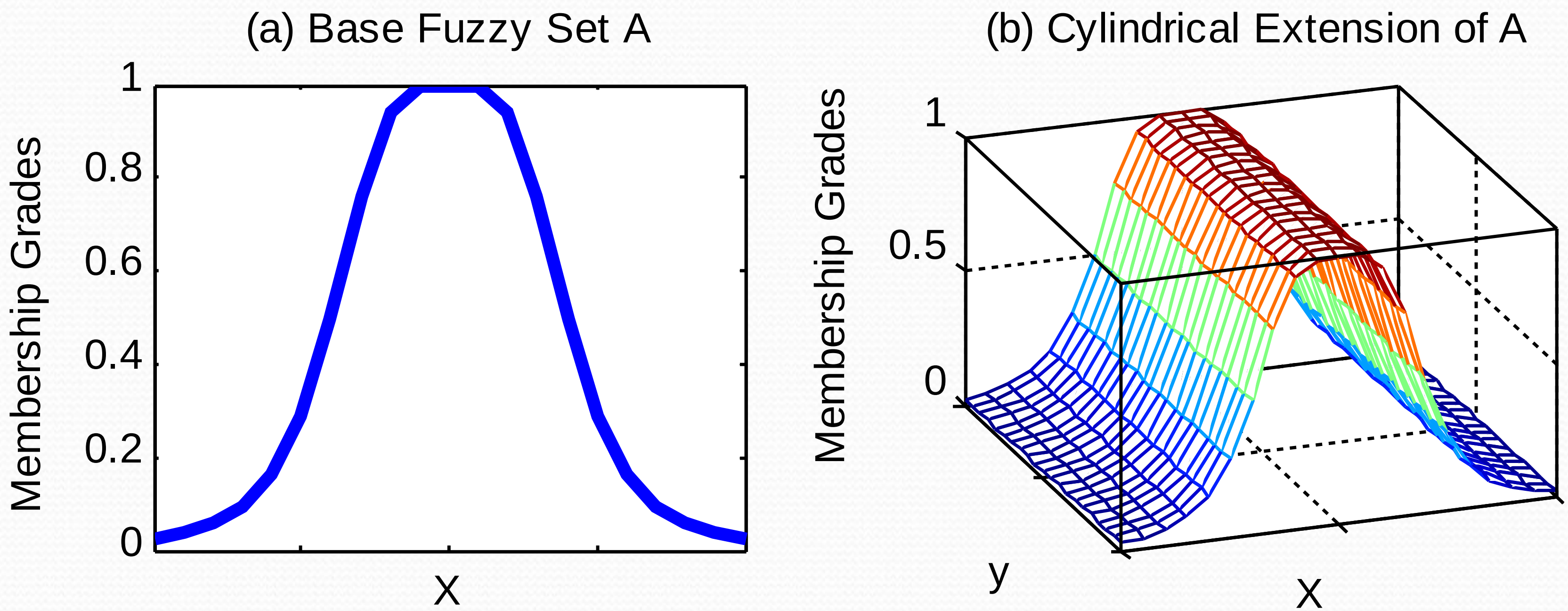
α=10

β=40

Cylindrical extension

The cylindrical extension of fuzzy set A in X into Y results in a two-dimensional fuzzy set in X x Y, and is given by

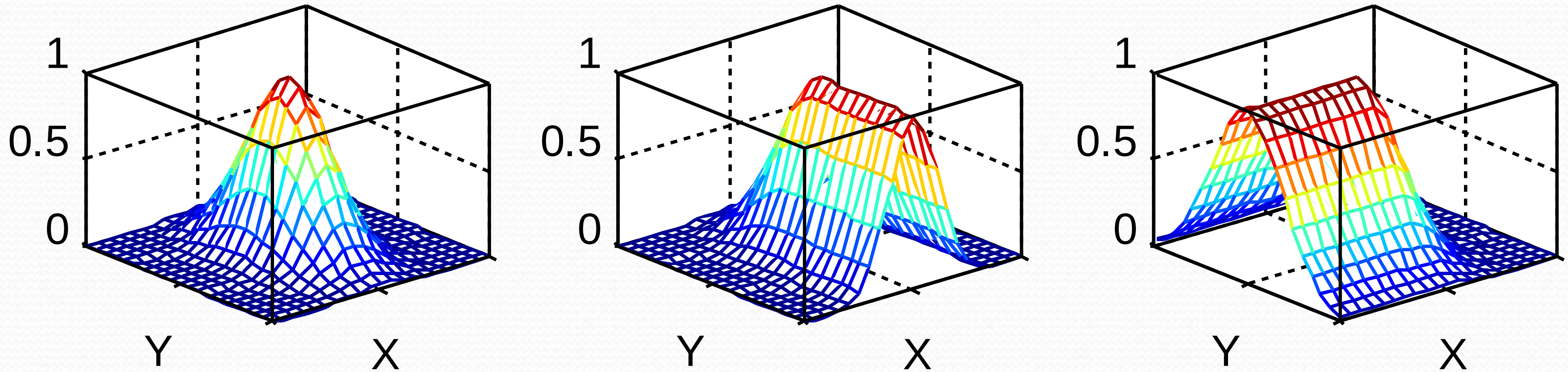
$$\mu_{CE_A}(x, y) = \mu_A(x), \forall y$$



Projection (shadow)

Two-dimensional MF | **Projection onto X** | **Projection onto Y**

(a) A Two-dimensional MF (b) Projection onto X (c) Projection onto Y

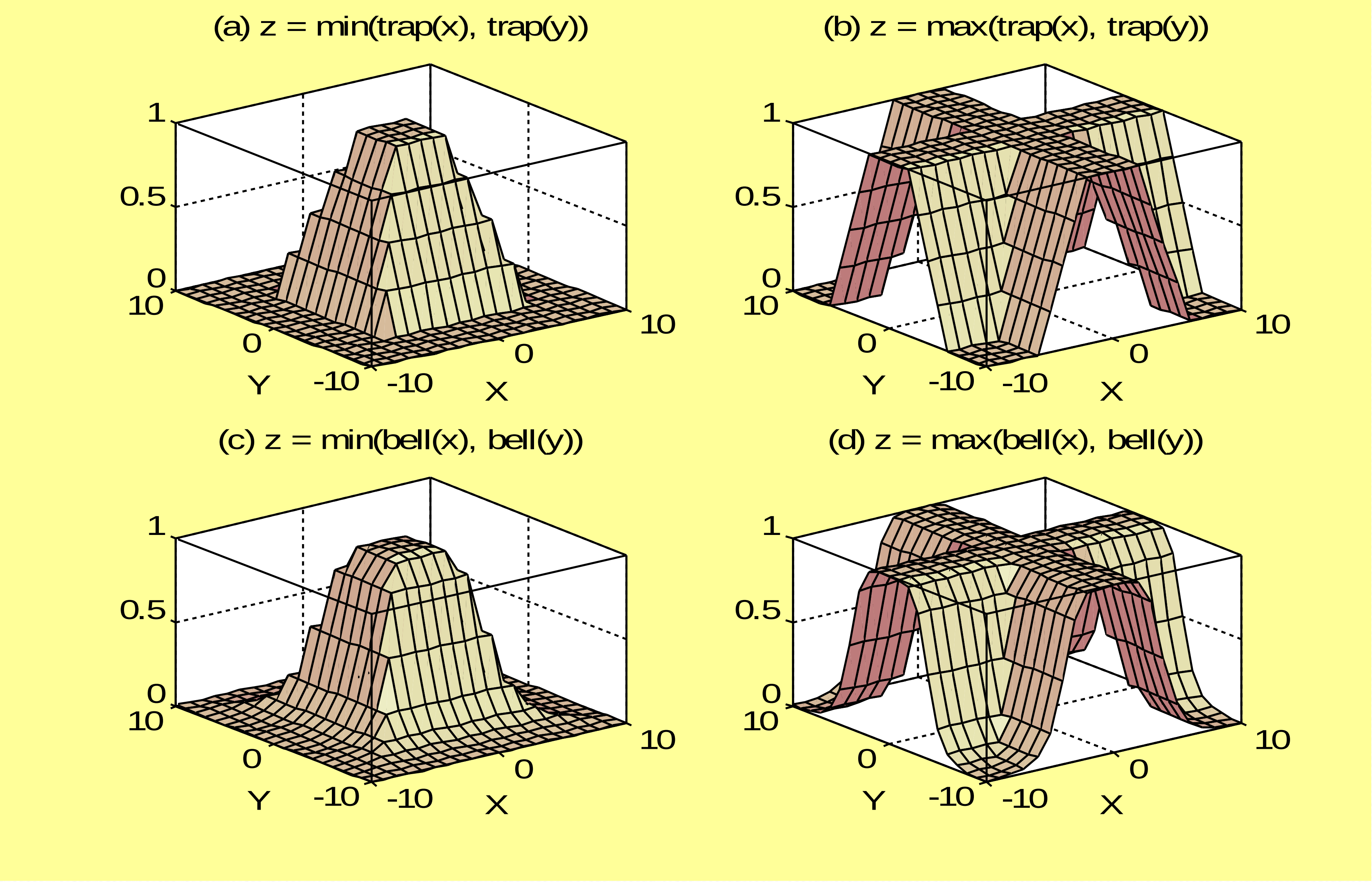


$$\mu_R(x, y)$$

$$\mu_A(x) = \max_y \mu_R(x, y)$$

$$\mu_B(y) = \max_x \mu_R(x, y)$$

2-D membership functions



Generalized negation

- General requirements:
  - Boundary: N(0)=1 and N(1) = 0
  - Monotonicity: N(a) > N(b) if a < b
  - Involution: N(N(a)) = a
- Two types of fuzzy complements:
  - Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

  - Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$

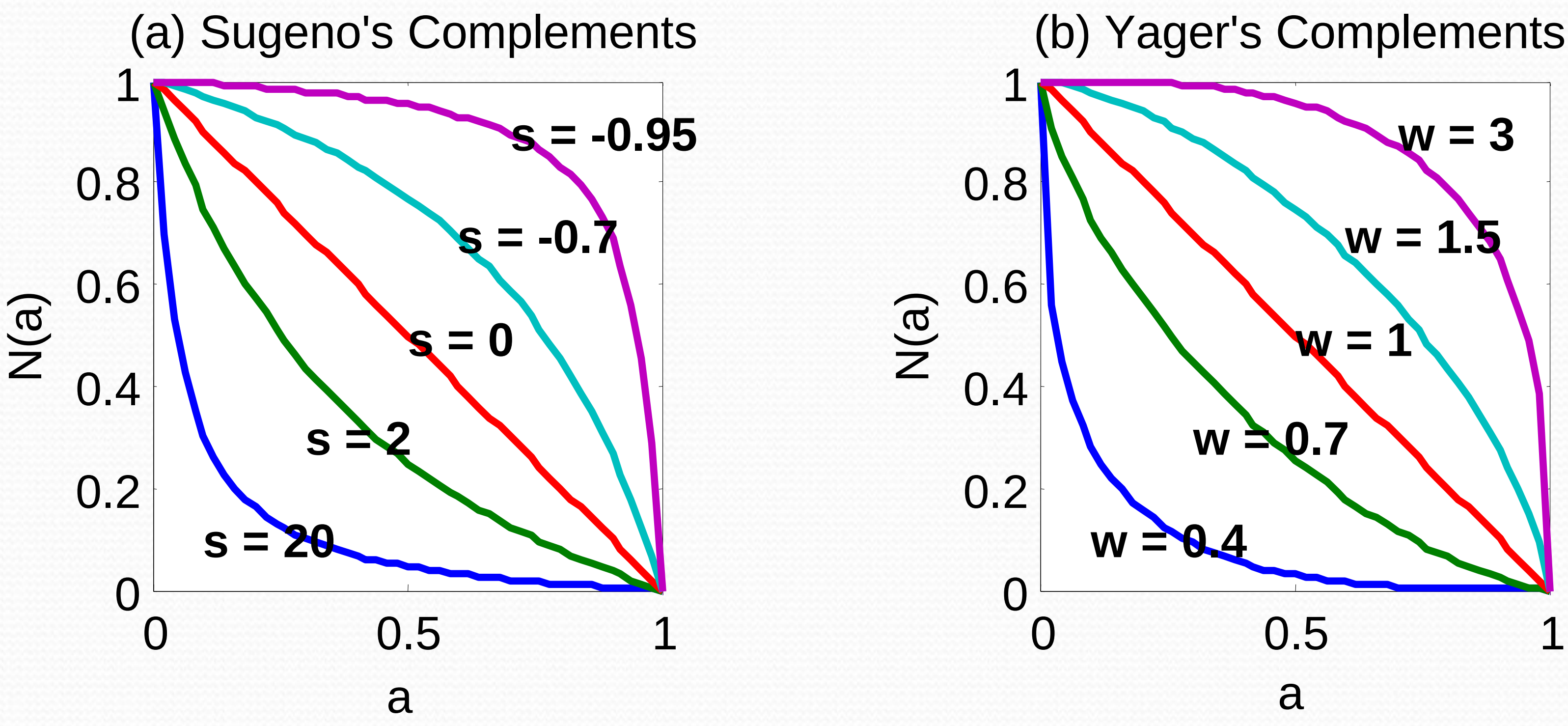
Sugeno's and Yager's complements

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



Generalized intersection (Triangular/T-norm)

- Basic requirements:
  - Boundary: T(0, a) = 0, T(a, 1) = T(1, a) = a
  - Monotonicity: T(a, b) <= T(c, d) if a <= c and b <= d
  - Commutativity: T(a, b) = T(b, a)
  - Associativity: T(a, T(b, c)) = T(T(a, b), c)

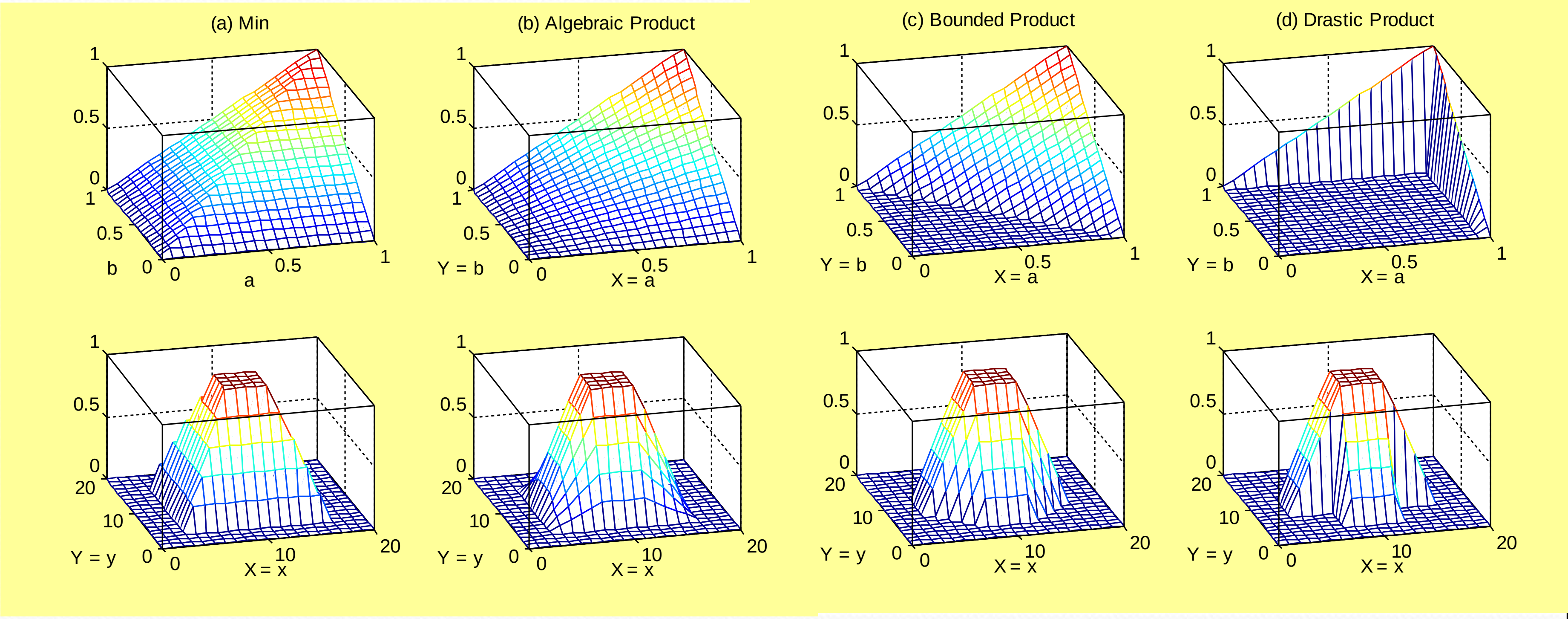
Generalized intersection (Triangular/T-norm)

- Examples:
  - Minimum: $T(a,b) = min(a,b) = a \wedge b$
  - Algebraic product: $T(a,b) = a \cdot b$
  - Bounded product: $T(a,b) = max(0,(a+b-1))$
  - Drastic product:

$$T(a,b) = \begin{cases} a & if\ b=1 \\ b & if\ a=1 \\ 0 & otherwise \end{cases}$$

T-norm operator

Minimum: $T_m(a, b)$ ≥ **Algebraic product: $T_a(a, b)$** ≥ **Bounded product: $T_b(a, b)$** ≥ **Drastic product: $T_d(a, b)$**

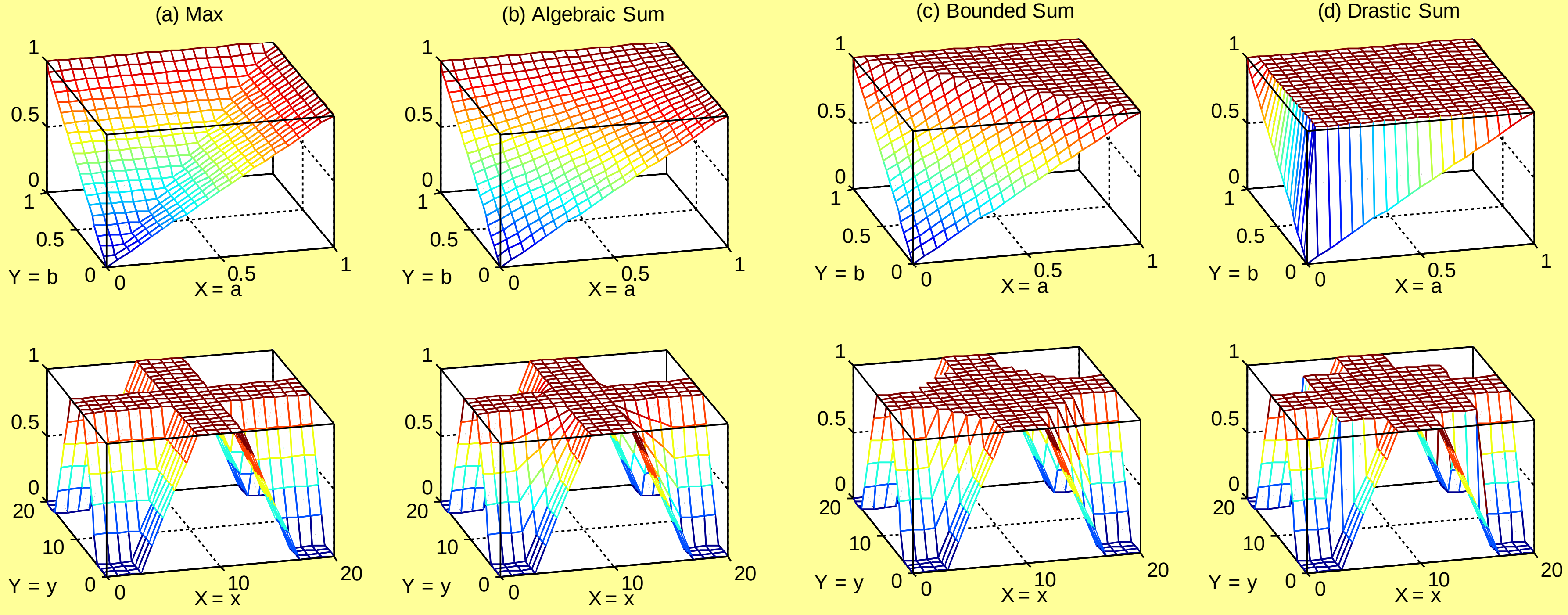


Generalized union (t-conorm)

- Basic requirements:
  - Boundary: S(1, a) = 1, S(a, 0) = S(0, a) = a
  - Monotonicity: S(a, b) < S(c, d) if a < c and b < d
  - Commutativity: S(a, b) = S(b, a)
  - Associativity: S(a, S(b, c)) = S(S(a, b), c)
- Examples:
  - Maximum: $S(a,b) = a \vee b$
  - Algebraic sum: $S(a,b) = a + b - a \cdot b$
  - Bounded sum: $S(a,b) = 1 \wedge (a + b)$
  - Drastic sum

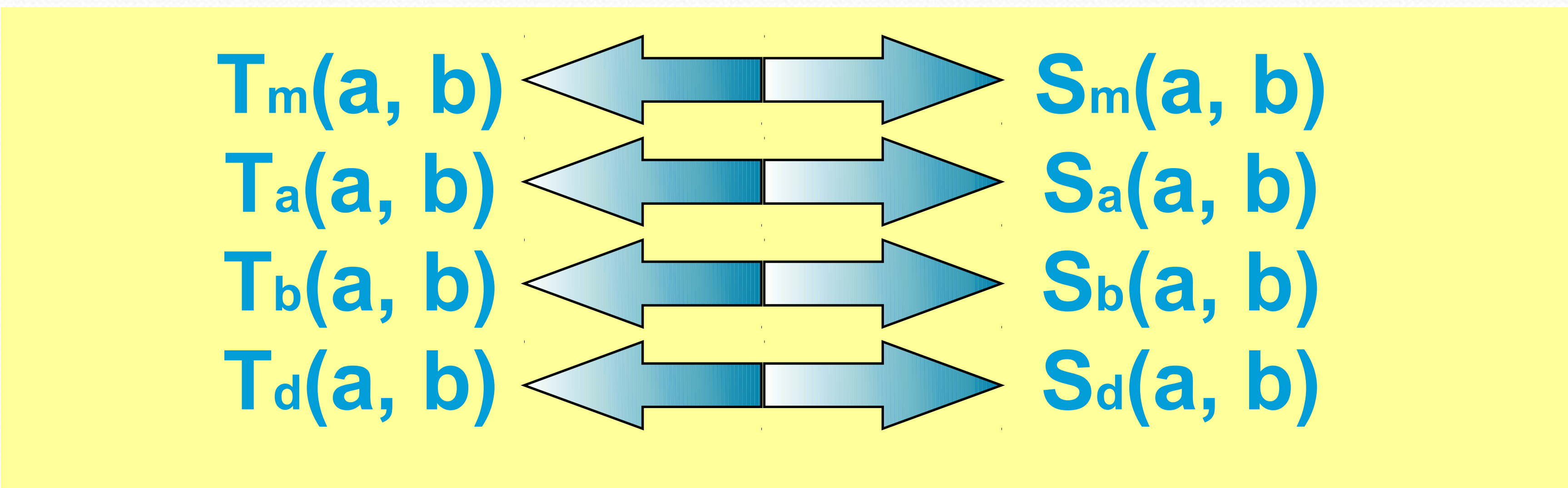
T-conorm operator

Maximum: $S_m(a, b)$ ≤ **Algebraic sum: $S_a(a, b)$** ≤ **Bounded sum: $S_b(a, b)$** ≤ **Drastic sum: $S_d(a, b)$**



Generalized De Morgan's Law

- T-norms and T-conorms are duals which support the generalization of DeMorgan's law:
  - $T(a, b) = N(S(N(a), N(b)))$
  - $S(a, b) = N(T(N(a), N(b)))$



Parameterized T-norm and S-norm

- Parameterized T-norms and dual T-conorms have been proposed by several researchers:
  - Yager
  - Schweizer and Sklar
  - Dubois and Prade
  - Hamacher
  - Frank
  - Sugeno
  - Dombi

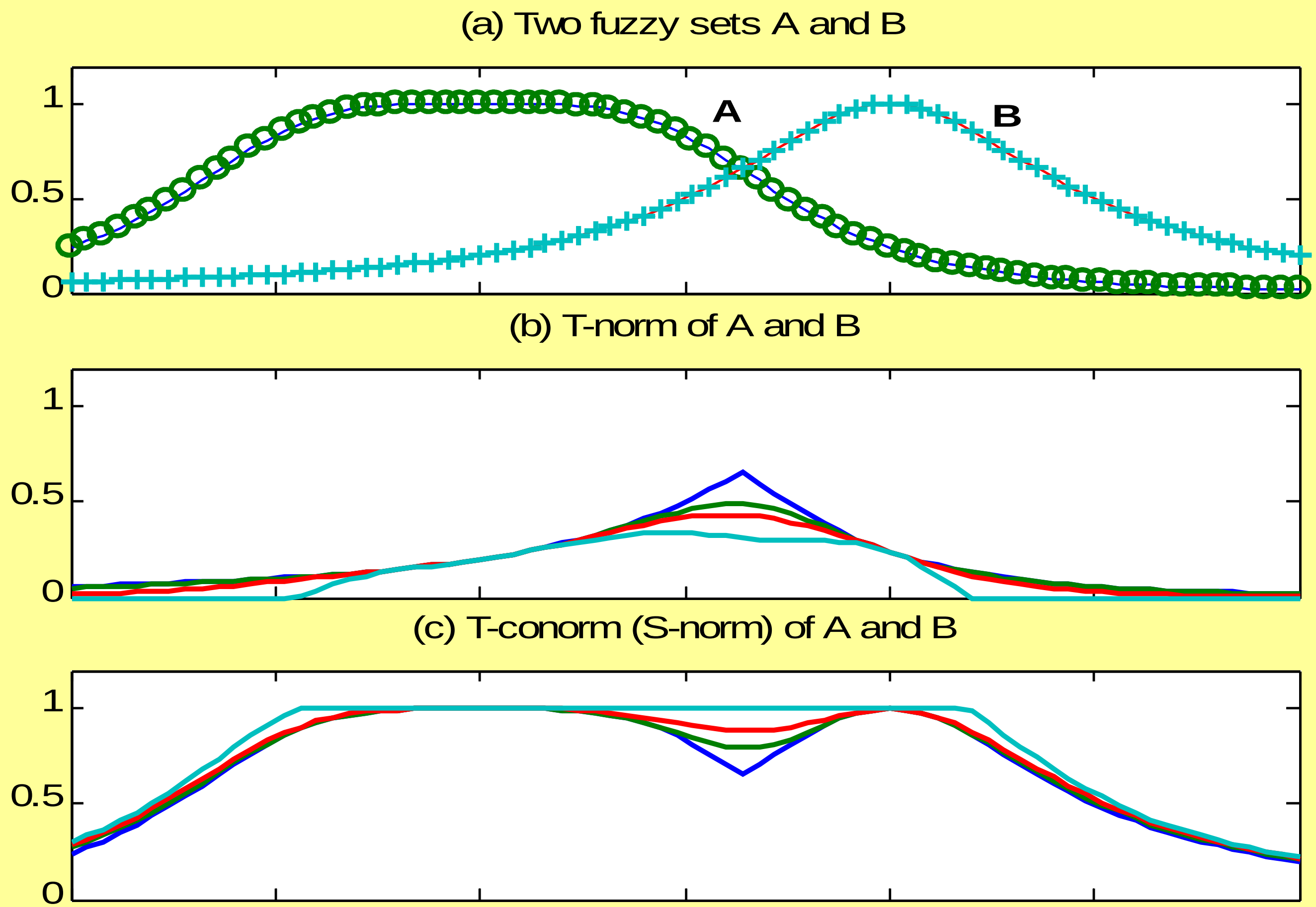
Schweizer and Sklar

$$\lim_{p\to 0}\left(\frac{f(p)}{f(0)}\right)^{\frac{1}{p}} = e^{\frac{f'(0)}{f(0)}}$$

$$T_{SS}(a,b,p) = [\max\{0,(a^{-p}+b^{-p}-1)\}]^{-\frac{1}{p}}$$

$$\lim_{p\to 0} T_{SS}(a,b,p) = ab$$

$$\lim_{p\to \infty} T_{SS}(a,b,p) = \min(a,b)$$



$$S_{SS}(a,b,p) = 1-[\max\{0,((1-a)^{-p}+(1-b)^{-p}-1)\}]^{-\frac{1}{p}}$$

Fuzzy relation

A fuzzy relation R between X and Y is a 2-D fuzzy subset of X x Y

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

with

$$\mu_R : X \times Y \rightarrow [0,1]$$

Examples:

- x is close to y
- x and y are similar
- x and y are related (dependent)

Discrete fuzzy relations

Relation: “is an important trade partner of”

	Holland	Germany	USA	Japan
Holland	1	0.9	0.5	0.2
Germany	0.3	1	0.4	0.2
USA	0.3	0.4	1	0.7
Japan	0.6	0.8	0.9	1

Max-min composition

The max-min composition of two fuzzy relations *R* (defined on *X* and *Y*) and *S* (defined on *Y* and *Z*) is

$$\mu_{R \circ S}(x, z) = \bigvee_{y} [\mu_R(x, y) \wedge \mu_S(y, z)]$$

The result is the combined relation defined on X and Z

Max-min composition

example

R **S** **R°S**

$$
\begin{bmatrix} 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8 \\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.9 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0.8 \\ 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0.2 \end{bmatrix}
$$

Max-product composition

The max-product composition of two fuzzy relations *R* (defined on *X* and *Y*) and *S* (defined on *Y* and *Z*) is

$$\mu_{R\circ S}(x,z)=\bigvee_{y}[\mu_R(x,y)\cdot\mu_S(y,z)]$$

The result is the combined relation defined on X and Z

Max-product composition

example

R ∘ **S** = **R°S**

$$
\begin{bmatrix}
0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\
0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\
0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8 \\
0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5 \\
0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2
\end{bmatrix}
\circ
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0.1 & 0.1 \\
0 & 0.2 & 0.2 \\
0 & 0.5 & 0.5 \\
0 & 0.5 & 0.9 \\
0 & 0.5 & 1
\end{bmatrix}
=
\begin{bmatrix}
0 & 0.5 & 1 \\
0 & 0.5 & 1 \\
0 & 0.4 & 0.8 \\
0 & 0.25 & 0.5 \\
0 & 0.1 & 0.2
\end{bmatrix}
$$

Extension principle

- A basic concept of fuzzy set theory
- General procedure to extend crisp mathematical expressions to fuzzy domains
- Generalizes a point-to-point mapping into a mapping between fuzzy sets
- Extends naturally to compositional rule of inference

Extension principle

A **is a fuzzy set on** ***X*** **:**

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

The image of ***A*** **under** ***f( )*** **is a fuzzy set** ***B:***

$$B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n$$

where $y_i = f(x_i)$, $i = 1$ **to** n**.**

If ***f( )*** **is a many-to-one mapping, then**

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$

Example

Let

$$A = \{(-2,0.1),(-1,0.4),(0,0.8),(1,0.9),(2,0.3)\}$$

and

$$f(x) = x^2 - 3$$

Then, computing membership functions gives

$$\hat{B} = \{(1,0.1),(-2,0.4),(-3,0.8),(-2,0.9),(1,0.3)\}$$

Consolidating, one obtains

$$B = \{(-3,0.8),(-2,0.4 \vee 0.9),(1,0.1 \vee 0.3)\}$$
$$= \{(-3,0.8),(-2,0.9),(1,0.3)\}$$

Definition

$$f : X_1 \times X_2 \times \cdots \times X_n \to Y$$

$$y = f(x_1, \ldots, x_n)$$

Suppose $A_1, \ldots, A_n$ are n fuzzy sets in $X_1, \ldots, X_n$

Then, fuzzy set B induced by f is given by

$$\mu_B(y) = \begin{cases} \max\limits_{\mathbf{x},\ \mathbf{x} = f^{-1}(y)} \ \min\limits_{i} \mu_{A_i}(x_i), \text{ if } f^{-1}(y) \neq \varnothing \\ 0, \text{ if } f^{-1}(y) = \varnothing \end{cases}$$

Continuous case